

DEPARTMENT OF MATHEMATICS NRU HSE
WRITTEN PH.D. ADMISSION EXAMINATION

October 17, 2012
(exam duration 5 hours)

1. Let f be a continuous function on the ray $[0, +\infty)$ such that f is differentiable on $(0, +\infty)$, its derivative strictly increases, and $f(0) = 0$. Prove that $g = f(x)/x$ also strictly increases on $(0, +\infty)$.

2. For which n does there exist an $n \times n$ matrix A with real coefficients satisfying the equation

$$A^2 + A + 7E = 0?$$

Here E denotes the identity $n \times n$ matrix.

3. Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{\sin n}{n}.$$

4. How many automorphisms does the group $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$ have?

5. A function is continuous in the closed unit disk $\{|z| \leq 1\}$ and is holomorphic in the interior of the disk. Can the image of the boundary of the disk be equal to the union of the unit circle $\{|z| = 1\}$ with the interval $[1, 2]$?

6. Show that there are no simple groups of order 30.

7. (a) Let X denote the union of two disjoint circles, and \mathbb{T} a two-dimensional torus (i.e. the direct product of two circles). Show that there are no continuous maps $f : X \rightarrow \mathbb{T}$ and $g : \mathbb{T} \rightarrow X$ such that $g \circ f$ is the identity map on X .

(b) Will the statement of part (a) remain true if the union of two disjoint circles is replaced by the figure 8 (i.e. by the union of two circles intersecting at one point).