Power and shift independent imaging of coherent sources

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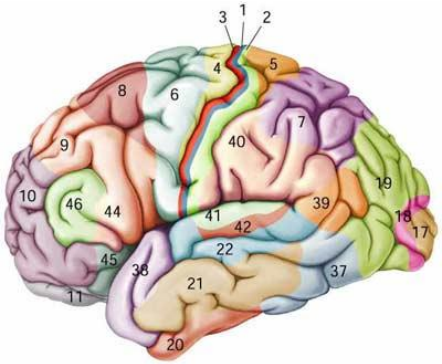
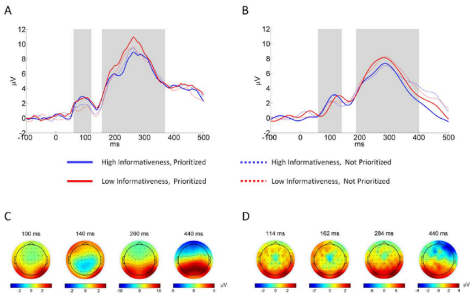
Studying brain activity...

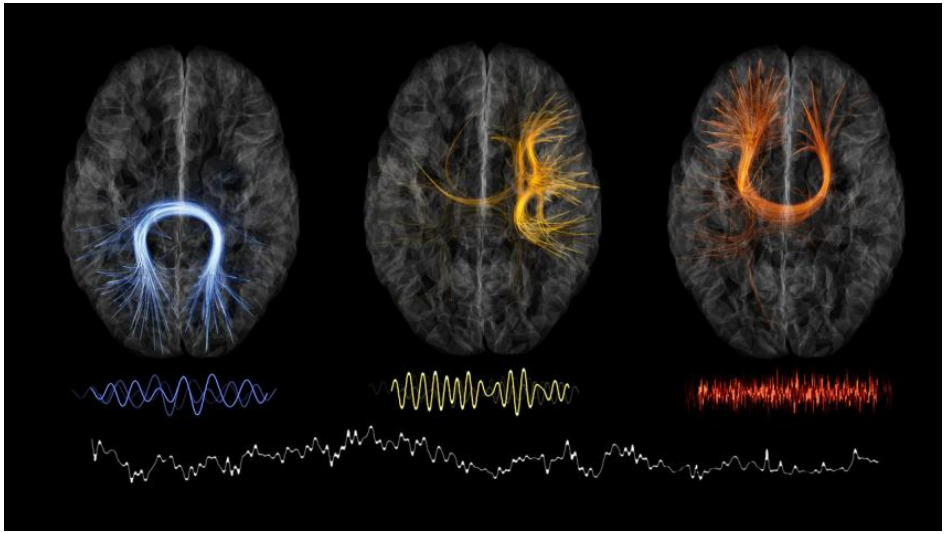
From evoked responses





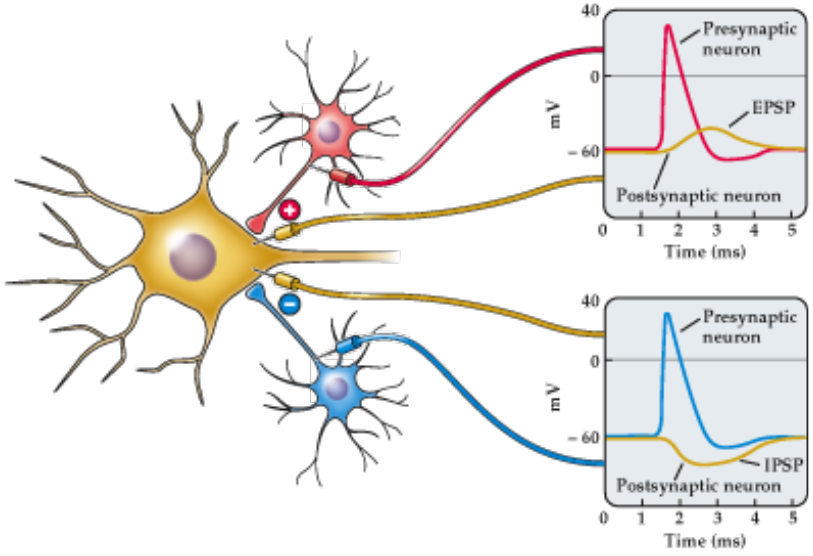
... to connectivity analysis

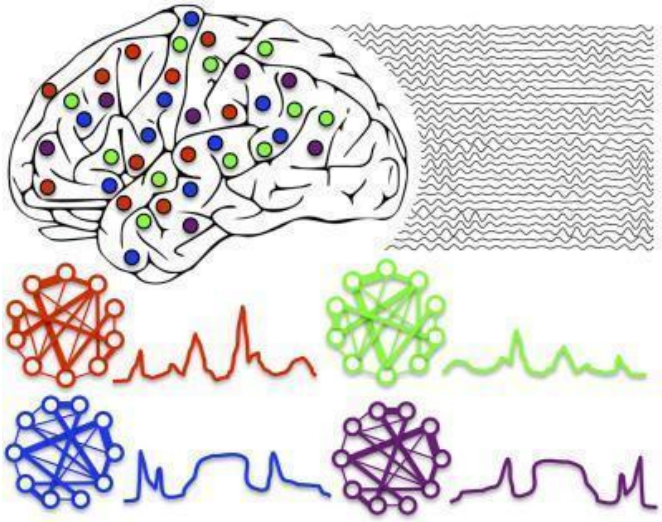
Functional / Effective

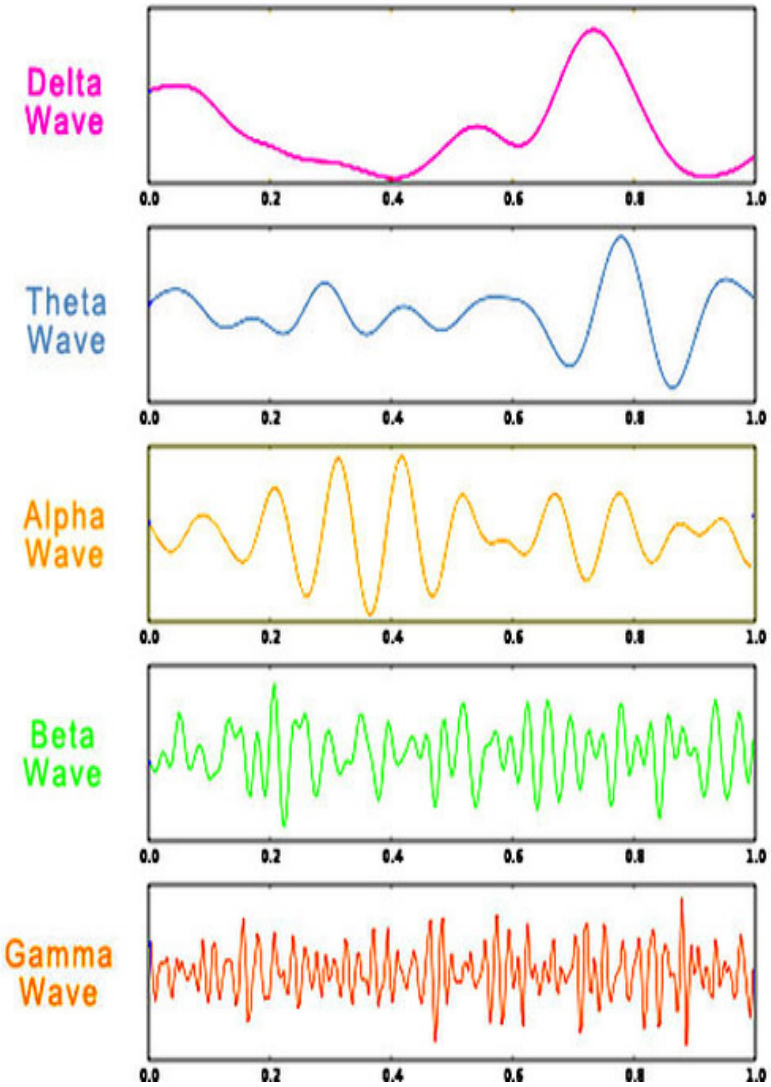


Anatomical

Connectivity = rhythms synchronization







Measuring brain activity

ECoG

invasive

fMRI

bad temporal resolution ($\approx$ 1s)

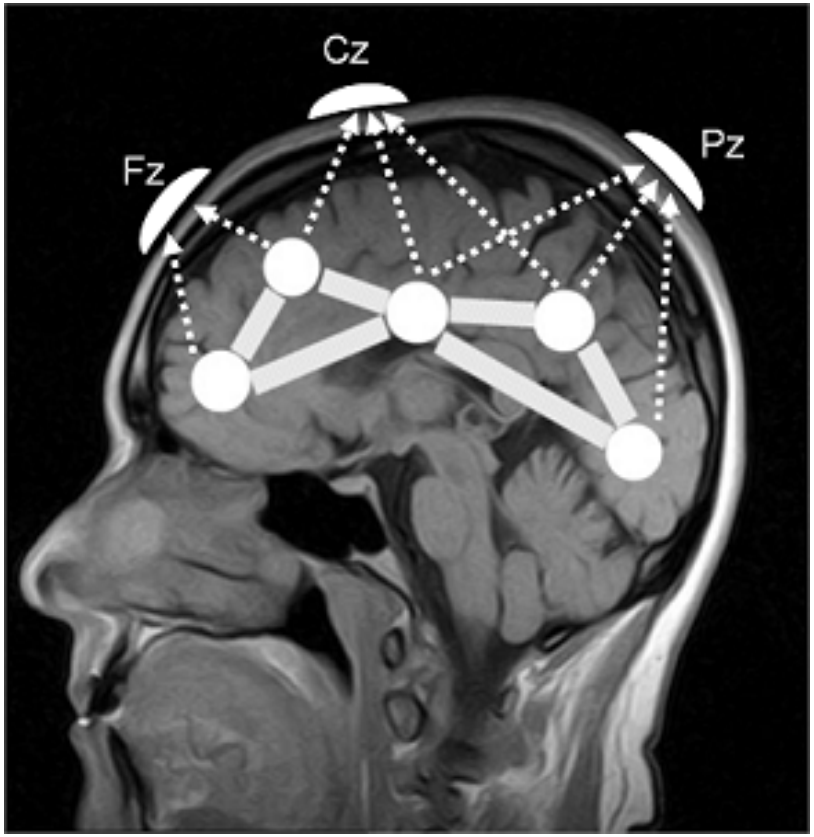
MEG

EEG

EEG and MEG allow to measure electromagnetic brain activity directly, noninvasively and with good temporal resolution

The idea of PSIICOS on a toy problem

Two problems:
volume conduction
and ill-posedness



The idea of PSIICOS on a toy problem

Generative model

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}(t) = \begin{pmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}(t) =$$

$$= \begin{pmatrix} g_1^1 \\ g_1^2 \end{pmatrix} s_1(t) + \begin{pmatrix} g_2^1 \\ g_2^2 \end{pmatrix} s_2(t) = \vec{g}_1 s_1(t) + \vec{g_2} s_2(t) \quad (1)$$

$m_{1,2}(t)$ - MEG/EEG measurements
$\{g_i^j\}$ - matrix of a forward model
$s_{1,2}(t)$ - unknown timeseries on cortex

The idea of PSIICOS on a toy problem

Time-frequency transformation

Apply time-frequency transform to (1)...

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}(f,t) = \vec{g}_1 S_1(f,t) + \vec{g_2} S_2(f,t) \tag{2}$$

... and write cross-spectrum:

$$\mathbf{C}^{MM}(t,f) \stackrel{def}{=} \mathbf{E}\{\mathbf{M}(t,f)\mathbf{M}^H(t,f)\} \tag{3}$$

N.B.

M_1, M_2, S_1, S_2 after time-frequency transformation are complex

Let's substitute (2) into (3)

$$\mathbf{C}^{MM}(t,f) = \mathbf{E}\left\{\begin{pmatrix} M_1\bar{M}_1 & M_1\bar{M}_2 \\ M_2\bar{M}_1 & M_2\bar{M}_2 \end{pmatrix}(f,t)\right\} =$$

$$\mathbf{E}\left\{\left(\vec{g}_1 S_1(f,t) + \vec{g_2} S_2(f,t)\right)\cdot\left(\vec{g}_1^T \bar{S}_1(f,t) + \vec{g_2}^T \bar{S}_2(f,t)\right)\right\} \quad (4)$$

Let's substitute (2) into (3)

$$\mathbf{C}^{MM}(t,f) = \mathbf{E}\left\{\begin{pmatrix} M_1\bar{M}_1 & M_1\bar{M}_2 \\ M_2\bar{M}_1 & M_2\bar{M}_2 \end{pmatrix}(f,t)\right\} =$$

$$\mathbf{E}\left\{\left(\vec{g}_1 S_1(f,t) + \vec{g}_2 S_2(f,t)\right)\cdot\left(\vec{g}_1^{\,T}\bar{S}_1(f,t) + \vec{g}_2^{\,T}\bar{S}_2(f,t)\right)\right\} \quad (4)$$

N.B.

$\vec{g}_i$ are real $\implies \vec{g}_i^{\,H} = \vec{g}_i^{\,T}$

Let's substitute (2) into (3)

$$\mathbf{C}^{MM}(t,f) = \mathbf{E}\left\{\begin{pmatrix} M_1\bar{M}_1 & M_1\bar{M}_2 \\ M_2\bar{M}_1 & M_2\bar{M}_2 \end{pmatrix}(f,t)\right\} =$$

$$\mathbf{E}\left\{\left(\vec{g}_1 S_1(f,t) + \vec{g}_2 S_2(f,t)\right)\cdot\left(\vec{g}_1^T \bar{S}_1(f,t) + \vec{g}_2^T \bar{S}_2(f,t)\right)\right\}$$

$$= \vec{g}_1\vec{g}_1^T\mathbf{E}\{S_1(f,t)\bar{S}_1(f,t)\} + \vec{g}_1\vec{g}_2^T\mathbf{E}\{S_1(f,t)\bar{S}_2(f,t)\} +$$

$$+ \vec{g}_2\vec{g}_1^T\mathbf{E}\{S_2(f,t)\bar{S}_1(f,t)\} + \vec{g}_2\vec{g}_2^T\mathbf{E}\{S_2(f,t)\bar{S}_2(f,t)\} \quad (4)$$

Let's substitute (2) into (3)

$$\begin{aligned}\mathbf{C}^{MM}(t,f) &= \\ &= \vec{g_1}\vec{g_1}^T\mathbf{E}\{S_1(f,t)\bar{S}_1(f,t)\} + \vec{g_1}\vec{g_2}^T\mathbf{E}\{S_1(f,t)\bar{S}_2(f,t)\}+ \\ &+ \vec{g_2}\vec{g_1}^T\mathbf{E}\{S_2(f,t)\bar{S}_1(f,t)\} + \vec{g_2}\vec{g_2}^T\mathbf{E}\{S_2(f,t)\bar{S}_2(f,t)\} \quad (4)\end{aligned}$$

Finally, we've got

$$\begin{pmatrix} c_{11}^{MM} & c_{12}^{MM} \\ c_{21}^{MM} & c_{22}^{MM} \end{pmatrix} = \\ = \vec{g_1}\vec{g_1}^T c_{11}^{SS} + \vec{g_1}\vec{g_2}^T c_{12}^{SS} + \vec{g_2}\vec{g_1}^T c_{21}^{SS} + \vec{g_2}\vec{g_2}^T c_{22}^{SS} \quad (5)$$

Or in matrix form:

$$\begin{pmatrix} c_{11}^{MM} & c_{12}^{MM} \\ c_{21}^{MM} & c_{22}^{MM} \end{pmatrix} =$$

$$= \begin{pmatrix} g_1^1 g_1^1 & g_1^1 g_1^2 \\ g_1^2 g_1^1 & g_1^2 g_1^2 \end{pmatrix} c_{11}^{SS} + \begin{pmatrix} g_1^1 g_2^1 & g_1^1 g_2^2 \\ g_1^2 g_2^1 & g_1^2 g_2^2 \end{pmatrix} c_{12}^{SS} +$$

$$+ \begin{pmatrix} g_2^1 g_1^1 & g_2^1 g_1^2 \\ g_2^2 g_1^1 & g_2^2 g_1^2 \end{pmatrix} c_{21}^{SS} + \begin{pmatrix} g_2^1 g_2^1 & g_2^1 g_2^2 \\ g_2^2 g_2^1 & g_2^2 g_2^2 \end{pmatrix} c_{22}^{SS} \quad (5)$$

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$$+ \begin{pmatrix} g_2^1 g_1^1 & g_2^1 g_1^2 \\ g_2^2 g_1^1 & g_2^2 g_1^2 \end{pmatrix} c_{21}^{SS} + \begin{pmatrix} g_2^1 g_2^1 & g_2^1 g_2^2 \\ g_2^2 g_2^1 & g_2^2 g_2^2 \end{pmatrix} c_{22}^{SS} \quad (5)$$

Volume conduction

We could have thrown the real part of this equation away [Nolte et al., 2004], but we can do better.

Or in matrix form:

$$\begin{pmatrix} c_{11}^{MM} & c_{12}^{MM} \\ c_{21}^{MM} & c_{22}^{MM} \end{pmatrix} =$$

$$= \begin{pmatrix} g_1^1 g_1^1 & g_1^1 g_1^2 \\ g_1^2 g_1^1 & g_1^2 g_1^2 \end{pmatrix} c_{11}^{SS} + \begin{pmatrix} g_1^1 g_2^1 & g_1^1 g_2^2 \\ g_1^2 g_2^1 & g_1^2 g_2^2 \end{pmatrix} c_{12}^{SS} +$$

$$+ \begin{pmatrix} g_2^1 g_1^1 & g_2^1 g_1^2 \\ g_2^2 g_1^1 & g_2^2 g_1^2 \end{pmatrix} c_{21}^{SS} + \begin{pmatrix} g_2^1 g_2^1 & g_2^1 g_2^2 \\ g_2^2 g_2^1 & g_2^2 g_2^2 \end{pmatrix} c_{22}^{SS} \quad (5)$$

Volume conduction

We could have thrown the real part of this equation away [Nolte et al., 2004], but we can do better.

Vectorized equation:

$$\begin{pmatrix} c_{11}^{MM} \\ c_{12}^{MM} \\ c_{21}^{MM} \\ c_{22}^{MM} \end{pmatrix} = \begin{pmatrix} g_1^1 g_1^1 \\ g_1^1 g_1^2 \\ g_1^2 g_1^1 \\ g_1^2 g_1^2 \end{pmatrix} c_{11}^{SS} + \begin{pmatrix} g_1^1 g_2^1 \\ g_1^1 g_2^2 \\ g_1^2 g_2^1 \\ g_1^2 g_2^2 \end{pmatrix} c_{12}^{SS} + \begin{pmatrix} g_2^1 g_1^1 \\ g_2^1 g_1^2 \\ g_2^2 g_1^1 \\ g_2^2 g_1^2 \end{pmatrix} c_{21}^{SS} + \begin{pmatrix} g_2^1 g_2^1 \\ g_2^1 g_2^2 \\ g_2^2 g_2^1 \\ g_2^2 g_2^2 \end{pmatrix} c_{22}^{SS}$$

Separating powers and interactions

$$\begin{pmatrix} c_{11}^{MM} \\ c_{12}^{MM} \\ c_{21}^{MM} \\ c_{22}^{MM} \end{pmatrix} = \begin{pmatrix} g_1^1 g_1^1 \\ g_1^1 g_1^2 \\ g_1^2 g_1^1 \\ g_1^2 g_1^2 \end{pmatrix} c_{11}^{SS} + \begin{pmatrix} g_1^1 g_2^1 \\ g_1^1 g_2^2 \\ g_1^2 g_2^1 \\ g_1^2 g_2^2 \end{pmatrix} c_{12}^{SS} + \begin{pmatrix} g_2^1 g_1^1 \\ g_2^1 g_1^2 \\ g_2^2 g_1^1 \\ g_2^2 g_1^2 \end{pmatrix} c_{21}^{SS} + \begin{pmatrix} g_2^1 g_2^1 \\ g_2^1 g_2^2 \\ g_2^2 g_2^1 \\ g_2^2 g_2^2 \end{pmatrix} c_{22}^{SS}$$

$\Uparrow$ $\Uparrow$

Let's project equation to the orthogonal complement of these vectors

$\mathbf{P}$:

$$\mathbf{F} = [vec(\mathbf{g_1 g_1^T}), vec(\mathbf{g_2 g_2^T})]$$
$$\mathbf{F} = \mathbf{USV^T}$$
$$\mathbf{U_2} = [\mathbf{u_1}, \mathbf{u_2}]$$
$$\mathbf{P} = \mathbf{I} - \mathbf{U_2 U_2^T}$$
$$\boldsymbol{C}^{\perp} = \boldsymbol{P} vec(\boldsymbol{C}^{\boldsymbol{MM}}) \tag{6}$$

Separating powers and interactions

Finally we get

$$\mathbf{P} \cdot \begin{pmatrix} c_{11}^{MM} \\ c_{12}^{MM} \\ c_{21}^{MM} \\ c_{22}^{MM} \end{pmatrix} (t) = \mathbf{P} \cdot \begin{pmatrix} g_1^1 g_2^1 \\ g_1^1 g_2^2 \\ g_1^2 g_2^1 \\ g_1^2 g_2^2 \end{pmatrix} c_{12}^{SS}(t) + \mathbf{P} \cdot \begin{pmatrix} g_2^1 g_1^1 \\ g_2^1 g_1^2 \\ g_2^2 g_1^1 \\ g_2^2 g_1^2 \end{pmatrix} c_{21}^{SS}(t)$$

We know

$$\mathbf{P}, \begin{pmatrix} c_{11}^{MM} \\ c_{12}^{MM} \\ c_{21}^{MM} \\ c_{22}^{MM} \end{pmatrix} (t), \begin{pmatrix} g_1^1 g_2^1 \\ g_1^1 g_2^2 \\ g_1^2 g_2^1 \\ g_1^2 g_2^2 \end{pmatrix}, \begin{pmatrix} g_2^1 g_1^1 \\ g_2^1 g_1^2 \\ g_2^2 g_1^1 \\ g_2^2 g_1^2 \end{pmatrix}$$

We want to find

$c_{12}^{SS}(t), c_{21}^{SS}(t)$

Global linear problem

We need to solve

$$vec(\mathbf{C}^{\perp}(t)) = \sum_{i=1}^{L}\sum_{j=1}^{L} vec(\mathbf{g_i g_j^T})^{\perp} c_{ij}^{ss}(t) + vec(\mathbf{C^{NN}}(t)) \tag{7}$$

$c_{ij}^{ss}(t)$ are the unknown timeseries on cortex which we are to recover

PSIICOS objection:

$$vec(\mathbf{C}^{\perp}(t)) = \sum_{i=1}^{L}\sum_{j=1}^{L} vec(\mathbf{g_i g_j^T})^{\perp} c_{ij}^{ss}(t) + vec(\mathbf{C^{NN}}(t)) \tag{8}$$

Define new variables:

Let $\mathbf{\Omega} = vec(\mathbf{C}^{\perp}(t))$,
$\mathbf{\Gamma_k} = vec(\mathbf{g_i g_j^T})^{\perp}$, $\sigma_{\mathbf{k}}(t) = c_{ij}^{ss}, \mathbf{N}(t) = \mathbf{C^{NN}}(t)$; then (8) will look like:

$$\mathbf{\Omega}(t) = \sum_{k=1}^{L^2} \mathbf{\Gamma_k} \sigma_{\mathbf{k}}(t) + \mathbf{N}(t) \tag{9}$$

MUSIC scan

(multiple signal classification, R. O. Schmidt, 1986)

One way to estimate coherent sources is to look at correlation of topographies with signal subspase

$$\mathbf{\Omega} = \mathbf{USV^T}$$

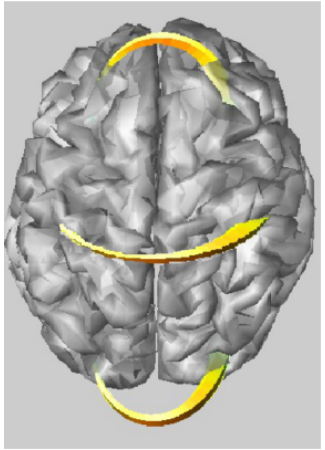
$$\mathbf{C_r} = [\mathbf{u_1}, \mathbf{u_2}, ..., \mathbf{u_r}]$$

$$A^{(n)} = \left\{ \sigma \quad \middle| \quad \left\| \mathbf{\Gamma_\sigma^T C_r} \right\| > threshold \right\}$$

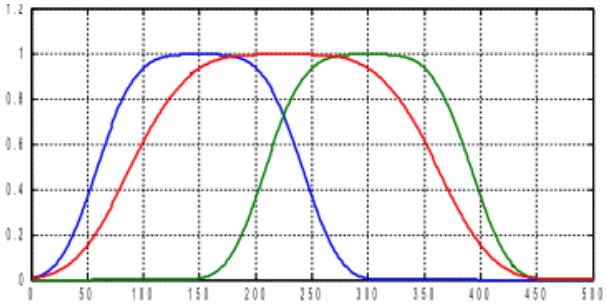
Simulations

- Three phase-locked networks
- Realistic brain noize ($1/f$ profile)
- Networks activity overlap in time

Sources locations:

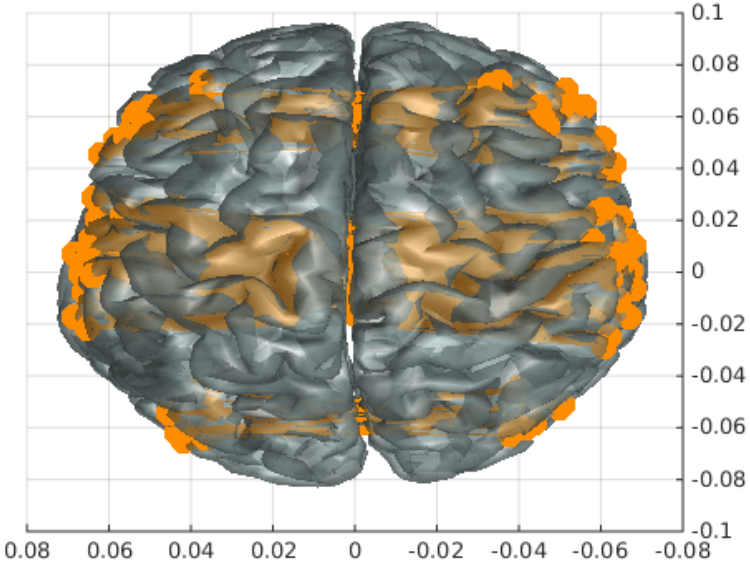


Synchrony profiles:

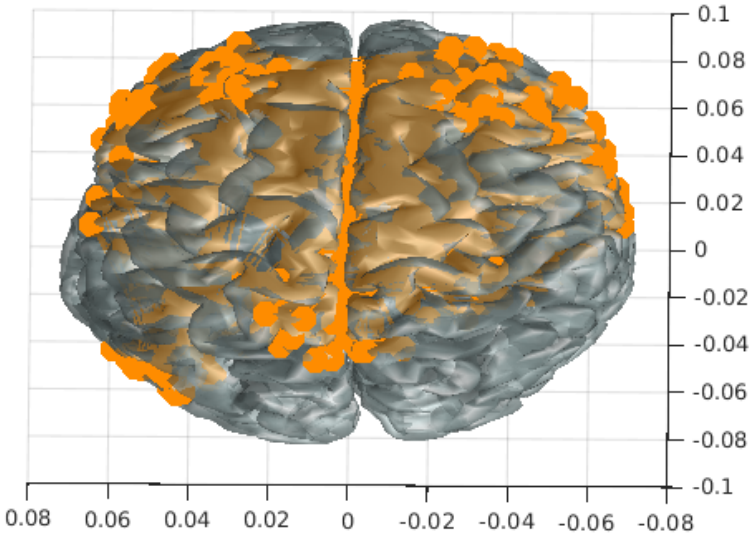


PSIICOS vs ImCoh

Source reconstruction
With VC-projection



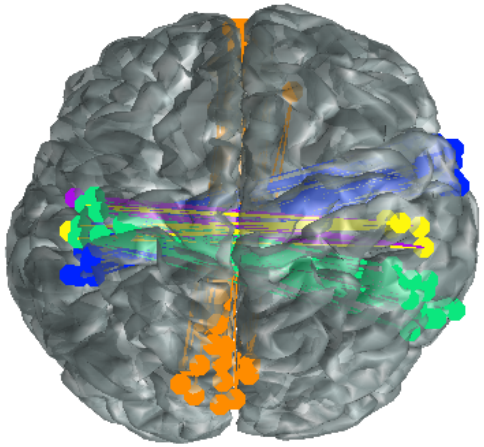
With imaginary cross-spectrum:



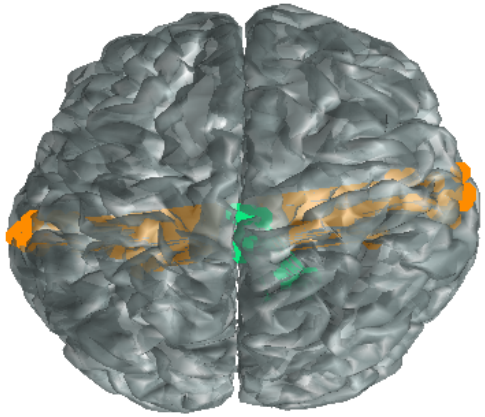
Real data scan

100 iterations bootstrap + Pairwise clustering [Zalesky, 2012]

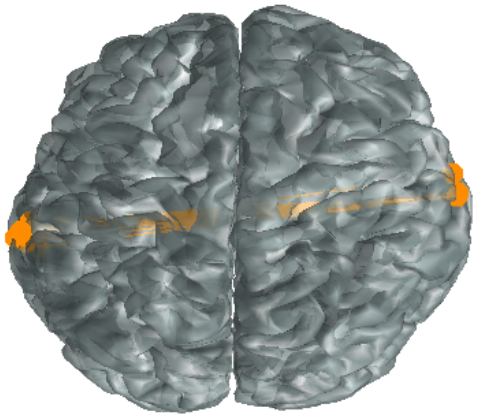
0003_pran, full, total, cond 2, 2-6 Hz, 0.4-0.7 sec



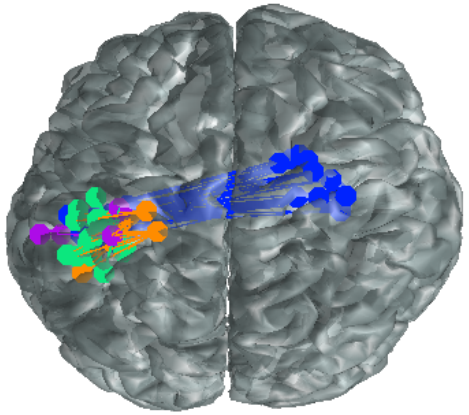
0003_pran, full, total, cond 2, 5-9 Hz, 0.4-0.7 sec



0003_pran, full, total,cond 2, 3-7Hz



0003_pran, full, total, cond 2, 19-23 Hz, 0.4-0.7 sec



Linear problem

So (9)

$$\mathbf{\Omega}(t) = \sum_{k=1}^{L^2} \mathbf{\Gamma_k} \sigma_{\mathbf{k}}(t) + \mathbf{N}(t)$$

Rewrites as

$$\frac{1}{2} \left\| \mathbf{\Omega}(t) - \sum_{k=1}^{L^2} \mathbf{\Gamma_k} \sigma_{\mathbf{k}}(t) \right\|_{Fro}^2 \longrightarrow min \tag{10}$$

Minimization problem

Linear problem

$$\frac{1}{2}\left\|\mathbf{\Omega}(t) - \sum_{k=1}^{L^2}\mathbf{\Gamma_k}\sigma_{\mathbf{k}}(t)\right\|_{Fro}^2 \longrightarrow min \tag{10}$$

Ill-posed problem! as $L^2 \approx 10^6$ and $\Omega(t)$ is $\approx 10^4 \times 1$, i.e. need to find $\approx 10^6$ unknowns from $\approx 10^4$ equations. Thus,

regularization is required

Linear problem

$$\frac{1}{2}\left\|\mathbf{\Omega}(t)-\sum_{k=1}^{L^2}\mathbf{\Gamma_k}\sigma_{\mathbf{k}}(t)\right\|_{Fro}^2+\lambda\sum_{k=1}^{L^2}\sqrt{\left\|\sigma_{\mathbf{k}}(t)\right\|}_{Fro}\longrightarrow min \quad (10)$$

Or in matrix notation

$$\frac{1}{2}\|\mathbf{\Omega}-\mathbf{\Gamma\Sigma}\|_{Fro}^2+\lambda\sum_{k=1}^{L^2}\sqrt{\|\mathbf{\Sigma}_k\|}_{Fro}\longrightarrow min \quad (11)$$

Mixed-norm regularization

Regularization term $\sum_{k=1}^{L^2} \sqrt{\|\mathbf{\Sigma}_k\|_{Fro}}$ has a $l_{2,0.5}$-norm penalty.

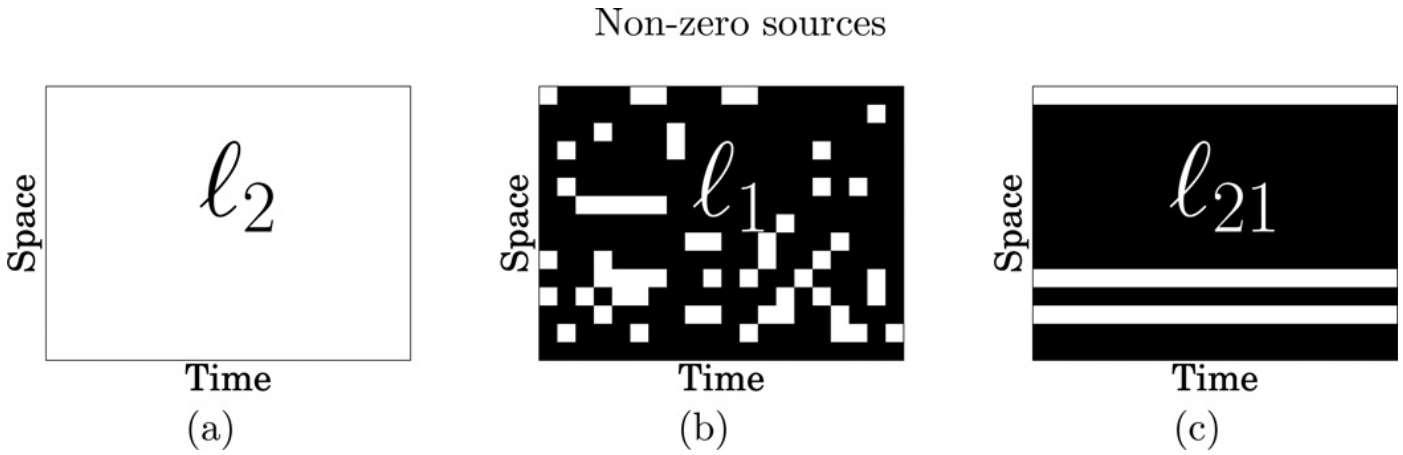


Figure: Solution structure for different regularization norms

Penalties are different for time and space directions in matrix S!!

Making problem convex...

$$\mathbf{\Sigma}^{(n)} = \underset{\mathbf{\Sigma}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{\Omega} - \mathbf{\Gamma}\mathbf{\Sigma}\|_{Fro}^2 + \lambda \sum_{k=1}^{L^2} \frac{\|\mathbf{\Sigma}_k\|_{Fro}}{2 \cdot \sqrt{\|\mathbf{\Sigma}_k^{(n-1)}\|_{Fro}}} =$$

$$= \underset{\mathbf{\Sigma}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{\Omega} - \mathbf{\Gamma}\mathbf{\Sigma}\|_{Fro}^2 + \lambda \sum_{k=1}^{L^2} \frac{1}{\mathbf{w_k^{(n)}}} \|\mathbf{\Sigma}_k\|_{Fro} \quad (12)$$

I.e. we approximate $l_{2,0.5}$-norm with weighted $l_{2,1}$-norm and get a convex problem. Similar approach is used in IRLS algorithm (Iterative Reweighted Least Squares)

Finally...

Ultimately, we get:

$$\mathbf{\Sigma}^{(n)} = \underset{\mathbf{\Sigma}}{\operatorname{argmin}} \frac{1}{2} \|\mathbf{\Omega} - \mathbf{\Gamma}\mathbf{W}^{\mathbf{n}}\mathbf{\Sigma}\|_{Fro}^2 + \lambda \sum_{k=1}^{L^2} \|\mathbf{\Sigma}_k\|_{Fro} =$$

$$= \underset{\mathbf{\Sigma}}{\operatorname{argmin}} \frac{1}{2} \left\|\mathbf{\Omega} - \mathbf{\Gamma}^{(\mathbf{n})}\mathbf{\Sigma}\right\|_{Fro}^2 + \lambda \sum_{k=1}^{L^2} \|\mathbf{\Sigma}_k\|_{Fro} \quad (13)$$

Outline of the algorithm

Repeat

1. Solve convex problem

$$\mathbf{\Sigma}^{(n)} = \underset{\mathbf{\Sigma}}{\operatorname{argmin}} \frac{1}{2} \left\| \mathbf{\Omega} - \mathbf{\Gamma}^{(\mathbf{n})} \mathbf{\Sigma} \right\|_{Fro}^{2} + \lambda \sum_{k=1}^{L^2} \| \mathbf{\Sigma}_k \|_{Fro}$$

   (we use Block-Coordinate Descent or BCD with duality gap stopping critereon)

2. Recalculate $\mathbf{\Gamma}^{(\mathbf{n})}$ based on $\mathbf{\Sigma}^{(n-1)}$

till convergence

So, we have $\approx 10^6 \times 10^4$ linear system for each timestep or $\approx 10^6 \times 10^4 \times 500$ in total. How to solve it?

Active set definition

Locations that are correlated with the residual error:

$$A^{(n)} = \left\{ \sigma \;\middle|\; \left\| \mathbf{\Gamma}_{\sigma}^{\mathbf{T}} (\mathbf{\Omega} - \mathbf{\Gamma}\mathbf{\Sigma}^{(\mathbf{n-1})}) \right\|_{Fro} > \lambda \right\} \tag{14}$$

Usage:

Pick locations from $A^{(n)}$; if solution is bad, expand $A^{(n)}$

Outline of the algorithm

Repeat

1. Solve convex problem on the active set

$$\mathbf{\Sigma_A}^{(n)} = \underset{\mathbf{\Sigma_A}}{\operatorname{argmin}} \frac{1}{2} \left\| \mathbf{\Omega} - \mathbf{\Gamma_A^{(n)}} \mathbf{\Sigma_A} \right\|_{Fro}^2 + \lambda \sum_{k_A=1}^{size(A)} \left\| \mathbf{\Sigma}_{k_A} \right\|_{Fro}$$

2. If duality gap is small, proceed to st. 3, if not, expand A and go to step 1
3. Recalculate $\mathbf{\Gamma^{(n)}}$ based on $\mathbf{\Sigma}^{(n-1)}$

till convergence

Simulated data

- 3 networks on real MRI grids
- Induced activity (same frequencies = 10 Hz, different envelopes for each network)
- Bandpass filter 2-20 Hz
- Constant phase shifts $(\pi/2, \pi/20)$ plus random phase supplement
- Solved on grid with 1503 verticies
- Artificial brain noise

Phase shift $= \frac{\pi}{2}$

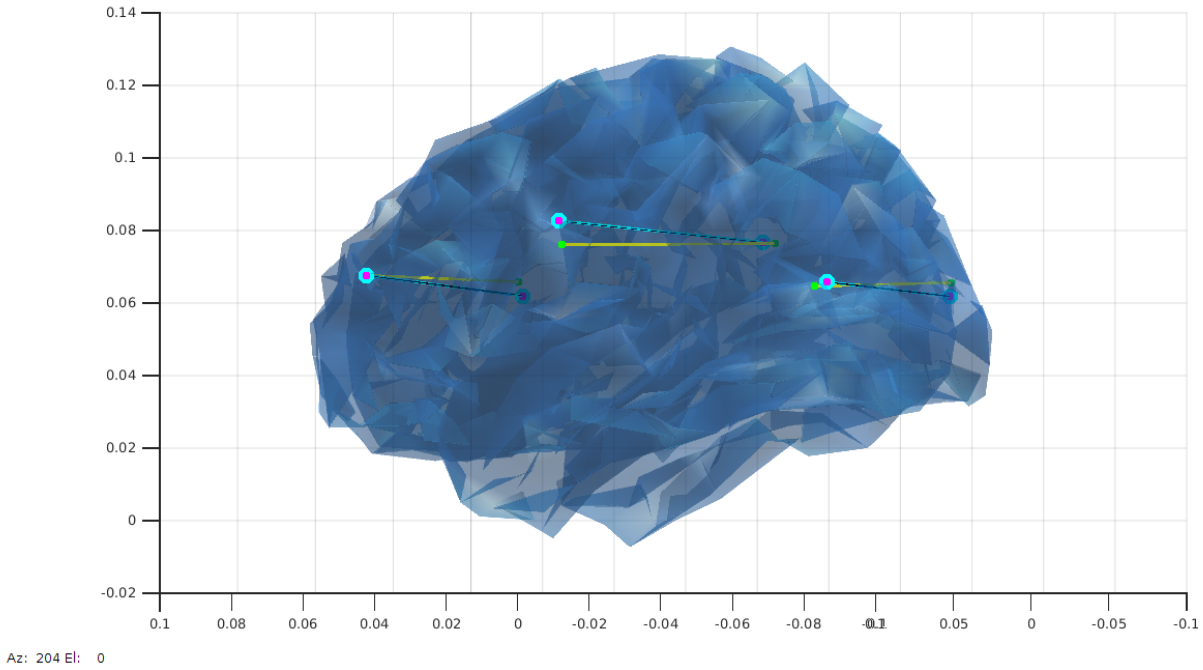


Figure: Source localizatios. Green - ground truth, cyan - solution

Phase shift $= \frac{\pi}{2}$

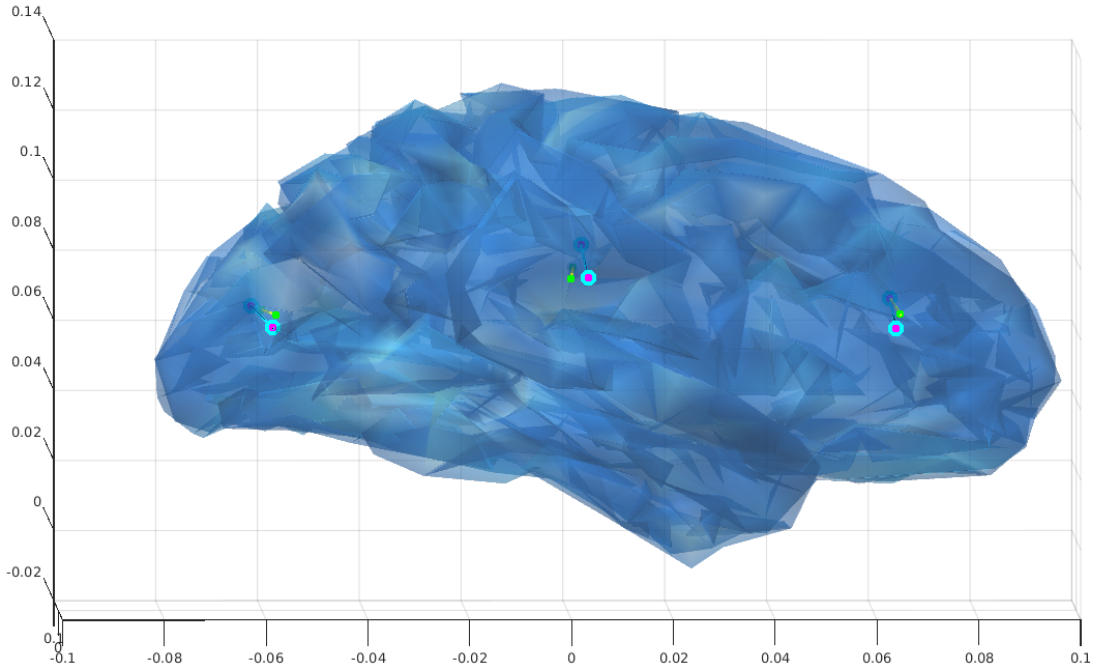


Figure: Source localizatios. Green - ground truth, cyan - solution

Phase shift $= \frac{\pi}{2}$

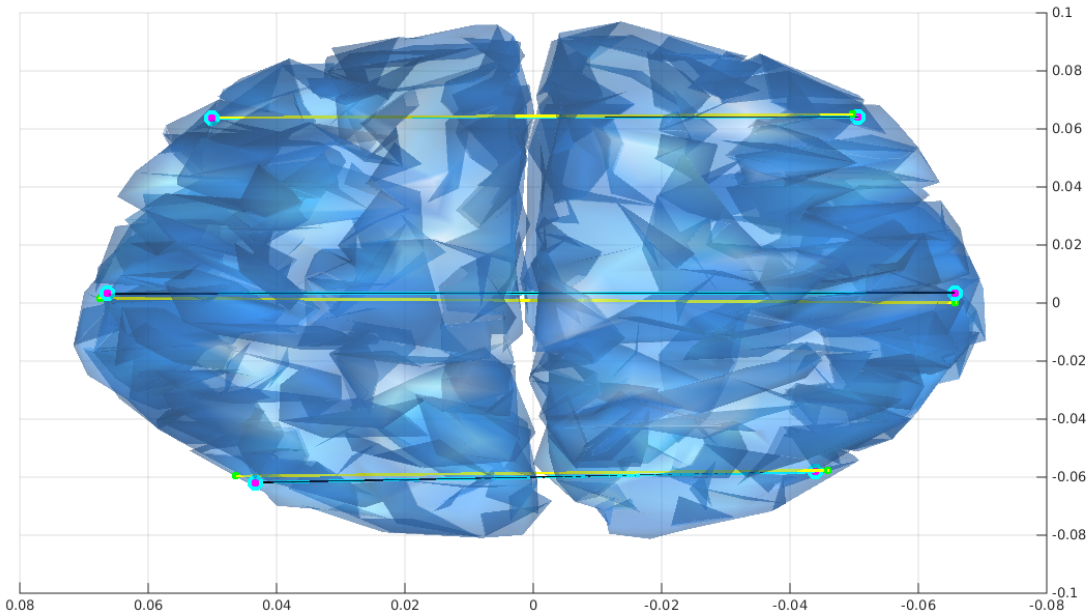
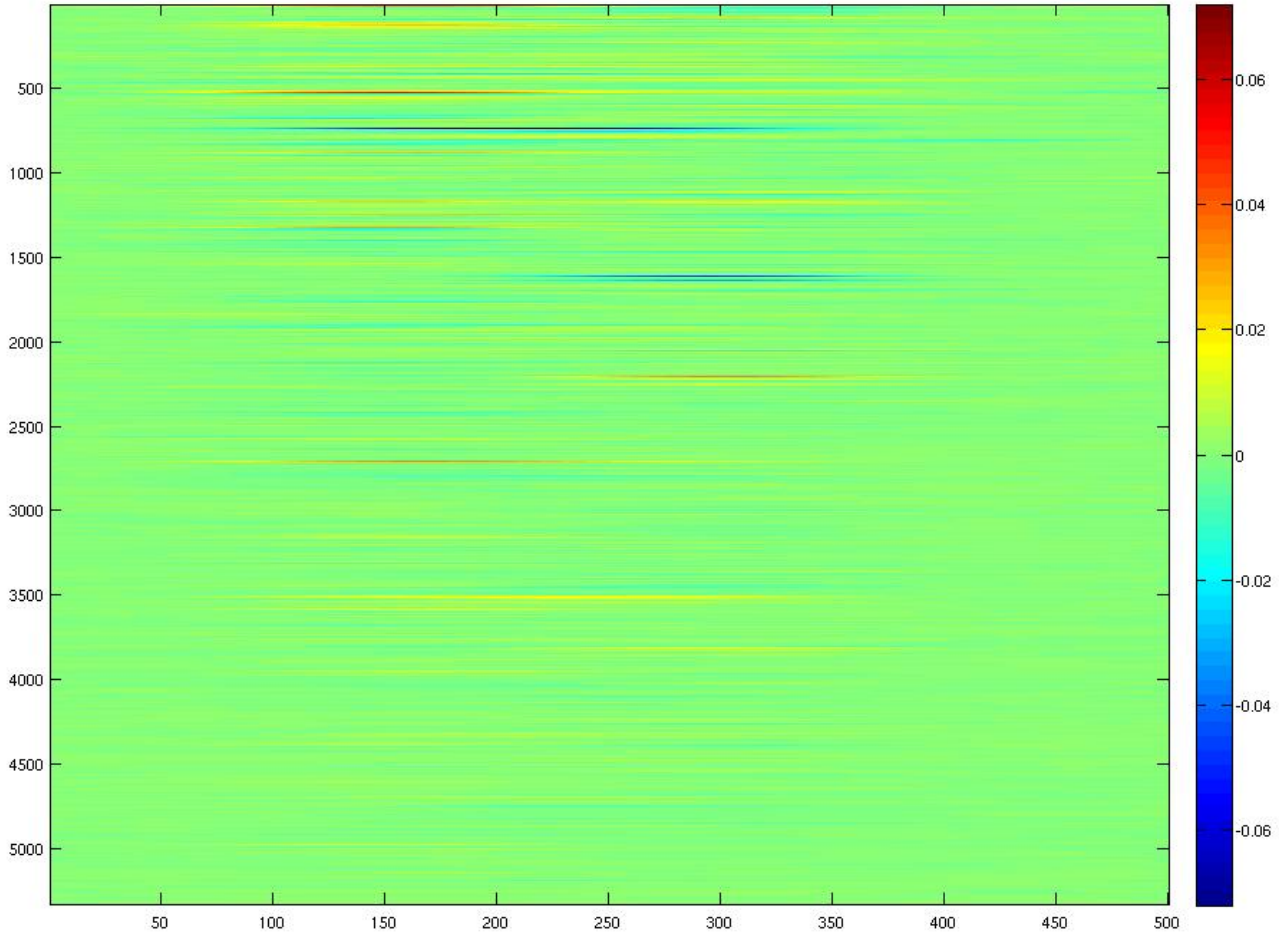
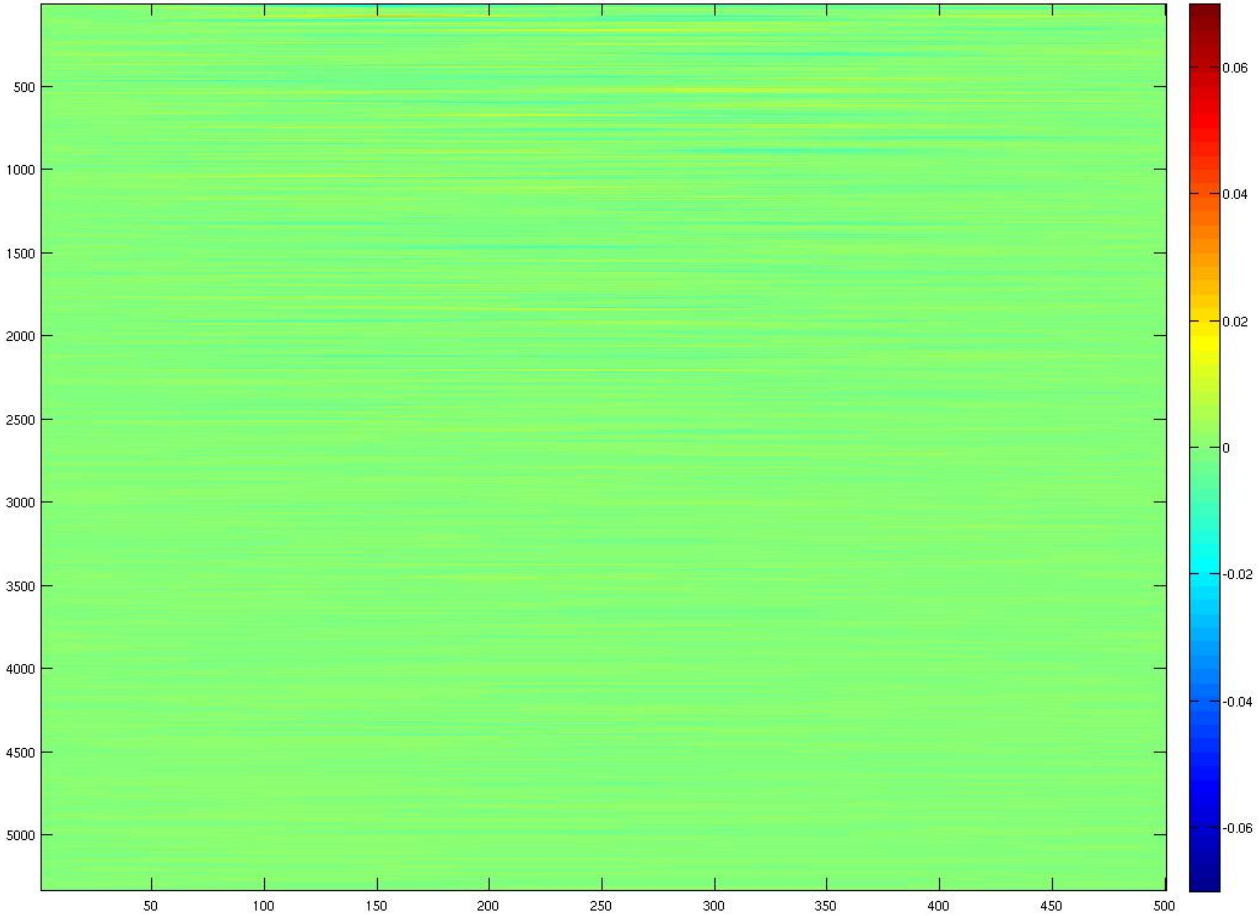


Figure: Source localizatios. Green - ground truth, cyan - solution

Measured cross-spectrum



Residual error



Phase shift $= \frac{\pi}{20}$

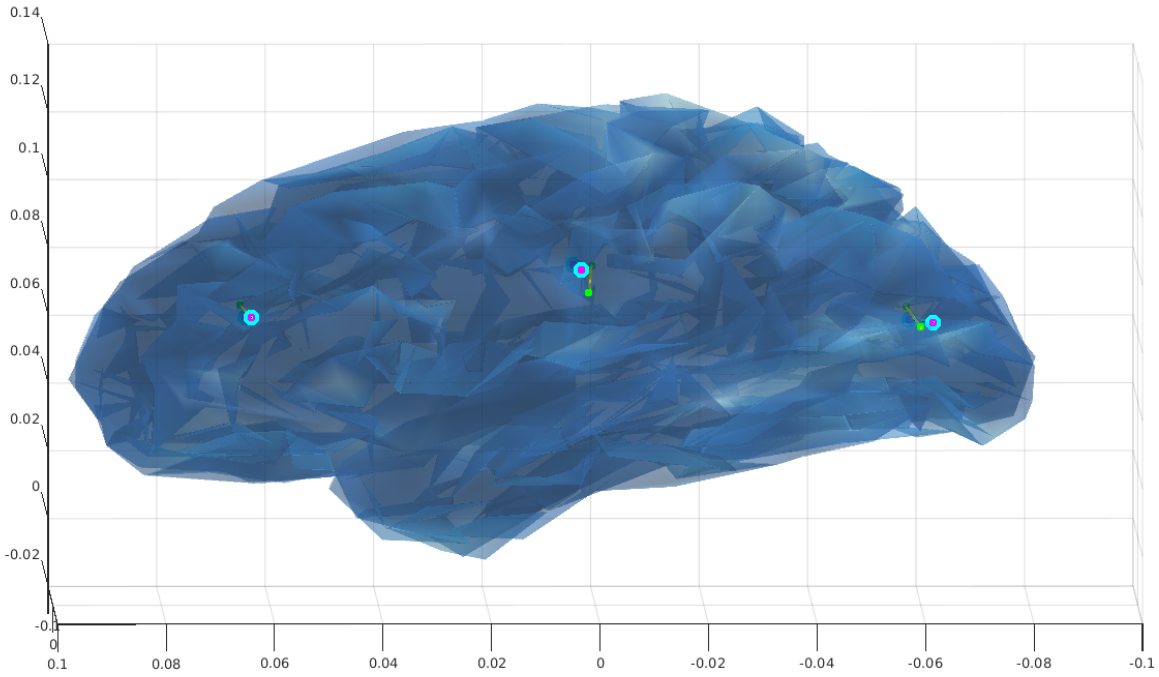


Figure: Source localizatios. Green - ground truth, cyan - solution

Phase shift $= \frac{\pi}{20}$

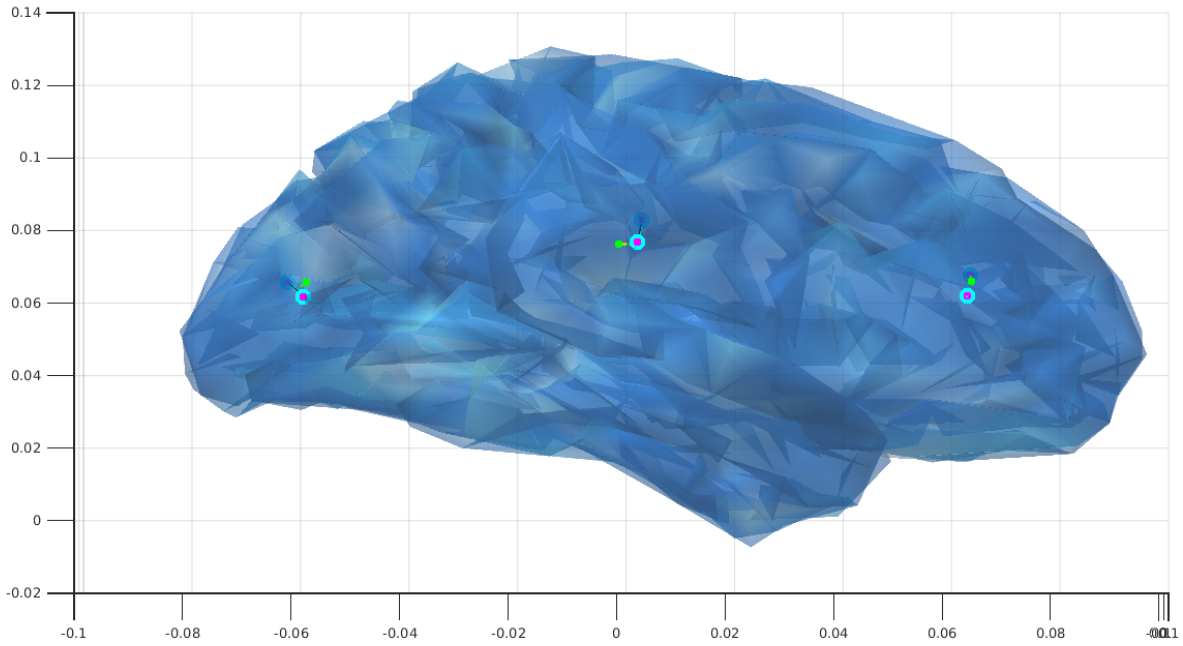


Figure: Source localizatios. Green - ground truth, cyan - solution

Phase shift $= \frac{\pi}{20}$

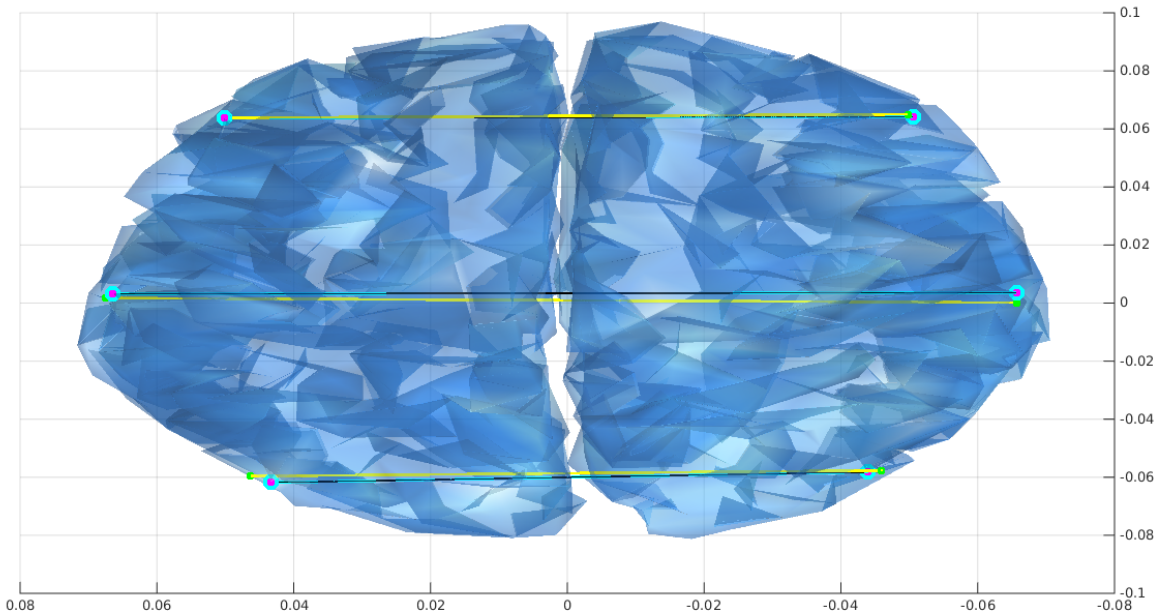


Figure: Source localizatios. Green - ground truth, cyan - solution

Real data

- Auditory odd-ball
- Movement-related words
- 120 trials
- 90 bootstrap runs
- Filtered in beta-band (16-25 Hz)

Real data; result after bootstrapping

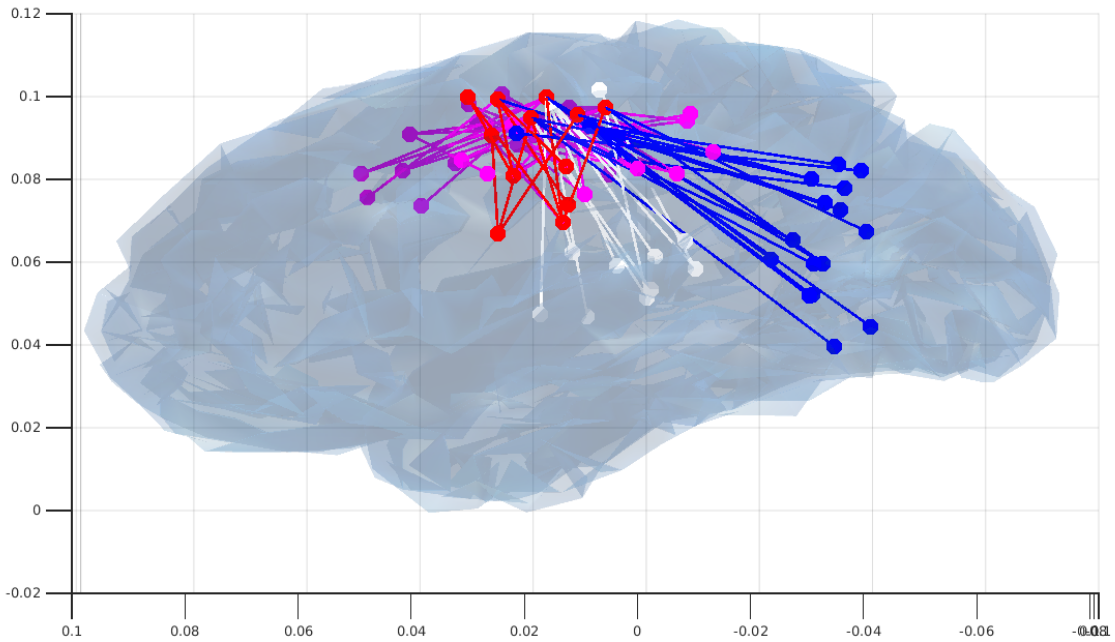


Figure: Sources localization

Real data; result after bootstrapping

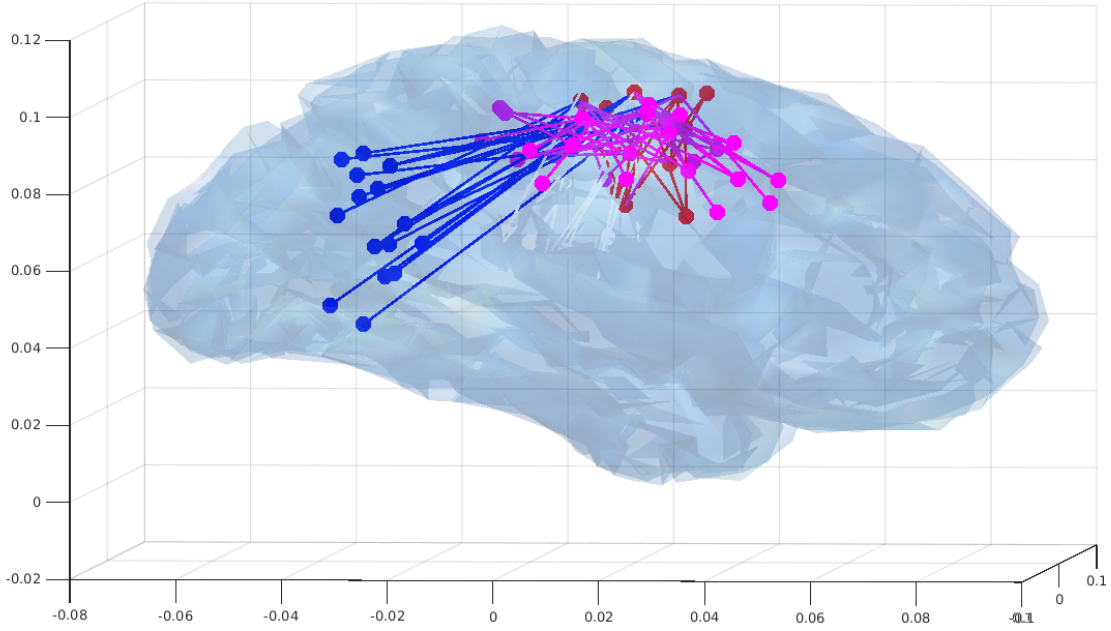


Figure: Sources localization

Real data; result after bootstrapping

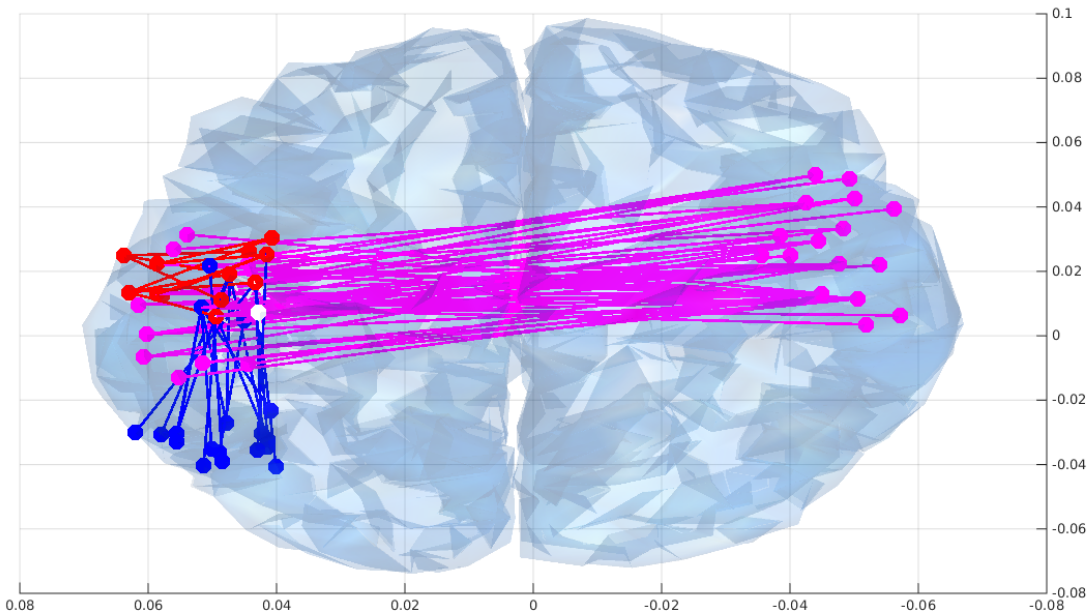


Figure: Sources localization

Thank you!