Weak conservativeness criterions for QGD-schemes for 1D gas dynamics equation

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Plan of the Presentation

The goal of our research is theoretical analysis of stability for a class of explicit difference schemes approximating gas dynamic equations.
This class of schemes is first of all linked to the names of B.N. Chetverushkin, T.G. Elizarova, Y.V. Sheretov, etc., and to transition to a regularized (quasi-gasdynamics) system of equations.
In this report, the system of equations for one-dimensional barotropic gas dynamics is considered. In addition to the sufficient condition for linearized stability (weak conservativeness) recently obtained by Y.V. Sheretov, the following is accomplished:

- A necessary spectral (von Neumann) condition for weak conservativeness is obtained
- A criterion (necessary and sufficient) condition for weak conservativeness is obtained
- Numerical experiments on weak conservativeness are done in the original nonlinear formulation.

1D barotropic quasi-gasdynamic (QGD)-system of equations has the following form:

$$\partial_t \rho + \partial_x j = 0, \quad \partial_t(\rho u) + \partial_x(ju + p(\rho) - \Pi) = 0, \tag{1}$$

$$j = \rho(u - w), \quad w = \frac{\tau}{\rho}\partial_x(\rho u^2) + \hat{w}, \quad \hat{w} = \frac{\tau}{\rho}[\rho u \partial_x u + p'(\rho)], \tag{2}$$

$$\Pi = \Pi_{NS} + \rho u \hat{w} + \tau p'(\rho)\partial_x(\rho u). \tag{3}$$

Here j and Π are regularized mass flow and stress, $\tau = \tau(\rho) > 0$ is a regularization parameter, and $\Pi_{NS} = \mu(\rho)\partial_x u$ is a viscous stress of Navier-Stokes, $\mu(\rho) \geq 0$ is proportional to the viscosity coefficient.

The system is linearized on a constant solution $\rho_* \equiv \mathrm{const} > 0$, $u_* = 0$. Substituting the solution in the form $\rho = \rho_* + \Delta\rho$, $u = u_* + \Delta u$ in the equations and neglecting the terms having at least second infinitesimal order with respect to $\Delta\rho$, Δu and their derivatives leads to the following system of equations:

$$\partial_t \Delta\rho + \rho_* \partial_x \Delta u = 0, \quad \rho_* \partial_t \Delta u + p'(\rho_*) \partial_x \Delta\rho u = 0. \qquad (4)$$

For dimensionless variables $\tilde{\rho} = \frac{\Delta\rho}{\rho_*}$, $\tilde{u} = \frac{\Delta u}{\sqrt{p'(\rho_*)}}$ we gain a system of equations of acoustics:

$$\partial_t \tilde{\rho} + \sqrt{p'(\rho_*)} \partial_x \tilde{u} = 0, \quad \partial_t \tilde{u} + \sqrt{p'(\rho_*)} \partial_x \tilde{\rho} = 0. \qquad (5)$$

Here $\sqrt{p'(\rho_*)}$ is a background velocity of sound. Given the initial values $\tilde{\rho}|_{t=0} = \tilde{\rho}_0$, $\tilde{u}|_{t=0} = \tilde{u}_0$ (that one may consider complex-valued) the conservation law holds for the last system:

$$\|\tilde{\rho}(\cdot,t)\|^2_{L^2(\mathbb{R})} + \|\tilde{u}(\cdot,t)\|^2_{L^2(\mathbb{R})} = \|\tilde{\rho}_0\|^2_{L^2(\mathbb{R})} + \|\tilde{u}_0\|^2_{L^2(\mathbb{R})} \quad \text{for} \quad t \geq 0. \qquad (6)$$

Let ω_h be a uniform mesh on $\mathbb{R}$ having nodes $x_k = kh$, $k \in \mathbb{Z}$ and step $h = X/N$. Let ω_h^* be an intermediate mesh having nodes $x_{k+1/2} = (k+0.5)h$, $k \in \mathbb{Z}$. Define a uniform mesh in t having nodes $t_m = m\Delta t$, $m \geq 0$ and step Δt. We define the shift, averaging and difference mesh operators

$$v_{\pm,k} = v_{k\pm 1}, \quad (sv)_{k-1/2} = \frac{v_k + v_{k+1}}{2}, \quad (\delta v)_{k-1/2} = \frac{v_k - v_{k-1}}{h},$$
$$(\delta^* y)_k = \frac{y_{k+1/2} - y_{k-1/2}}{h}, \quad \delta_t v = \frac{\hat{v} - v}{\Delta t}, \quad \hat{v}^m = v^{m+1}.$$

We consider a standard two-level explicit and three-point symmetric in space discretization of equations:

$$
\begin{aligned}
&\frac{\tilde{\rho}-\rho}{\Delta t}+\delta^* j=0,\\
&\frac{\tilde{\rho}\tilde{u}-\rho u}{\Delta t}+\delta^*(j\cdot su+s\cdot p(\rho)-\Pi)=0,\\
&j=(s\rho)su-(s\rho)w,\\
&(s\rho)w=\tilde{\tau}[\delta(\rho u)]su+(s\rho)\hat{w},\\
&(s\rho)\hat{w}=\tilde{\tau}[(s\rho)(su)\delta u+\delta p(\rho)],\\
&\Pi=\mu\delta u+(su)(s\rho)\hat{w}+\tau(sp'(\rho))\delta(\rho u).
\end{aligned}
$$

Linearized difference scheme

We linearize the scheme on a constant solution $\rho_* \equiv \mathrm{const} > 0$, $u_* = 0$. We write its solution in the form $\rho = \rho_* + \Delta\rho$, $u = u_* + \Delta u$ and neglect terms that have the second infinitesimal order with respect to $\Delta\rho$, Δu. So we have

$$\delta_t \Delta\rho + \rho_* \delta^* s \Delta u - \tau(\rho_*) p'(\rho_*) \delta^* \delta \Delta\rho = 0,$$
$$\rho_* \delta_t \Delta u + p'(\rho_*) \delta^* s \Delta\rho - \left[\mu(\rho_*) + \tau(\rho_*) \rho_* p'(\rho_*)\right] \delta^* \delta \Delta u = 0.$$

For dimensionless variables $\tilde{\rho} = \frac{\Delta\rho}{\rho_*}$, $\tilde{u} = \frac{\Delta u}{c_*}$ we have the equations

$$\delta_t \tilde{\rho} + c_* \delta^* s \tilde{u} - \tau(\rho_*) c_*^2 \delta^* \delta \tilde{\rho} = 0, \tag{7}$$
$$\delta_t \tilde{u} + c_* \delta^* s \tilde{\rho} - \left[\frac{\mu(\rho_*)}{\rho_*} + \tau(\rho_*) c_*^2\right] \delta^* \delta \tilde{u} = 0, \tag{8}$$

Linearized difference scheme

We assume that the regularization parameter and viscosity coefficient are evaluated by the formulae

$$\tau(\rho) = \frac{\alpha h}{\sqrt{p'(\rho)}}, \quad \mu(\rho) = \alpha_s \tau(\rho) \rho p'(\rho),$$

where $\alpha > 0$ and $\alpha_s \geq 0$ are parameters. Then (omitting tildes above ρ, u), the equations (7), (8) may be rewritten in the following recurrent form

$$\hat{\rho} = \rho - \frac{\beta}{2}(u_+ - u_-) + \alpha\beta(\rho_- - 2\rho + \rho_-), \tag{9}$$

$$\hat{u} = u - \frac{\beta}{2}(\rho_+ - \rho_-) + \varkappa\alpha\beta(u_+ - 2u + u_-) \tag{10}$$

with three parameters α, $\beta := c_* \frac{\Delta t}{h}$ and $\varkappa := \alpha_s + 1 \geq 1$. The functions ρ^0 and u^0 are given, i. e., we consider the initial-value (Cauchy) problem for the scheme. Further it will be convenient for us to consider ρ and u as complex-valued functions.

Weak conservativeness analysis

We consider a column-vector function $\mathbf{y}^m = (\rho^m \ u^m)^T$, $m \geq 0$ and rewrite the linearized difference scheme (9), (10) in the matrix form

$$\hat{\mathbf{y}} = \begin{pmatrix} \alpha\beta & \frac{\beta}{2} \\ \frac{\beta}{2} & \varkappa\alpha\beta \end{pmatrix} \mathbf{y}_- + \begin{pmatrix} 1-2\alpha\beta & 0 \\ 0 & 1-2\varkappa\alpha\beta \end{pmatrix} \mathbf{y} + \begin{pmatrix} \alpha\beta & -\frac{\beta}{2} \\ -\frac{\beta}{2} & \varkappa\alpha\beta \end{pmatrix} \mathbf{y}_+.$$

Let H be a Hilbert space of complex-valued square-summable vector functions on ω_h, i. e., having a finite norm

$$\|\mathbf{y}\|_H = \left(h \sum_{k=-\infty}^{\infty} |\mathbf{y}_k|^2\right)^{1/2}.$$

Given $\mathbf{y}^0 = (\rho^0 \ u^0)^T \in H$, we have $\mathbf{y}^m \in H$ for any $m \geq 1$. We say that the scheme (10) is *weakly conservative* if the following estimate holds:

$$\sup_{m \geq 0} \|\mathbf{y}^m\|_H \leq \|\mathbf{y}^0\|_H \quad \forall \mathbf{y}^0 \in H. \tag{11}$$

Our numerical experiments show that namely this property corresponds well to numerical solutions without the spurious oscillations for our class of schemes.

We substitute a solution of the form $\mathbf{y}_k^m = e^{\mathbf{i}k\xi}\mathbf{v}^m(\xi)$, $k \in \mathbb{Z}$, $m \geq 0$, in (10), where $\mathbf{i}$ is the imaginary unit, $0 \leq \xi \leq 2\pi$, so we get

$$\hat{\mathbf{v}}(\xi) = G(\xi)\mathbf{v}(\xi), \quad G(\xi) = \begin{pmatrix} 1-\omega_1 & -\mathbf{i}\omega_2 \\ -\mathbf{i}\omega_2 & 1-\varkappa\omega_1 \end{pmatrix}, \tag{12}$$

where we define $\omega_1 = 4\alpha\beta\theta$, $\theta = \sin^2\frac{\xi}{2} \in [0,1]$, $\omega_2 = \beta\sin\xi$; we emphasize that $\omega_2^2 = 4\beta^2\theta(1-\theta)$.
It can be proven that given $\mathbf{y}^0 = (\rho^0\ u^0)^T \in H$ we can define a function $\mathbf{v}^0 \in L^2(0,2\pi)$ such that

$$\mathbf{v}^0(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \mathbf{y}_k^0 e^{-\mathbf{i}k\xi},$$

and write the solution of the scheme (10) in the integral form

$$\mathbf{y}_k^m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \mathbf{v}^m(\xi) e^{\mathbf{i}k\xi}\, d\xi, \quad k \in \mathbb{Z},$$

where $\mathbf{v}^m \in L^2(0,2\pi)$ by (12). The Parseval equality also holds:

$$\|\mathbf{y}^m\|_H = \sqrt{h}\,\|\mathbf{v}^m\|_{L^2(0,2\pi)}, \quad m \geq 0. \tag{13}$$

It is known that the condition

$$\max_{0\le\xi\le 2\pi} \max_{l} |\lambda_l\big(G(\xi)\big)| \le 1 \tag{14}$$

is the *necessary spectral condition* for (11) to hold. Hereafter $\lambda_l(A)$ are eigenvalues of matrix A.

Lemma 1

The inequality

$$\max_{0\le\xi\le 2\pi} \max_{l} \lambda_l\big((G^* G)(\xi)\big) \le 1 \tag{15}$$

is a necessary and sufficient condition for (11) *to hold.*

The following results are received:

Theorem 2

The necessary condition (14) *holds if and only if*

$$\beta \leq \min\left\{(\varkappa + 1)\alpha, \frac{1}{2\varkappa\alpha}\right\}. \tag{16}$$

Theorem 3

The criterion (15) *holds if and only if*

$$\beta \leq \min\left\{2\alpha, \frac{1}{2\varkappa\alpha}\right\}. \tag{17}$$

Theorem 4

For simplified quasi-hydrodynamical regularization where we omit the terms with $\delta(\rho u)$ *in the original scheme and* $\tau\rho_* c_*^2$ *in the linearized one:*
1) in the case $0 \le \alpha_s \le 1$ *the necessary condition* (14) *and the criterion* (15) *hold if and only if*

$$\beta \le \min\left\{(\alpha_s + 1)\alpha, \frac{1}{2\alpha}\right\}, \tag{18}$$

$$\beta \le \min\left\{2\alpha_s\alpha, \frac{1}{2\alpha}\right\}; \tag{19}$$

respectively.
2) in the case $\alpha_s \ge 1$ *the results of the theorems 2 and 3 hold with* $\varkappa = \alpha_s$.

The following figure presents necessary spectral condition, criterion for weak conservativeness and the results of numerical experiments with parameters $\gamma = 2$ (the shallow water equations), $p_1 = 1, \kappa = \frac{7}{3}$. Consider the Riemann problem on the time interval $0 \le t \le 0.5$ with initial data:

$$\rho_0(x) = \begin{cases} 1, & x < 0 \\ 0.1, & x > 0 \end{cases}, \quad u_0(x) = \begin{cases} 0.1, & x < 0 \\ 0, & x > 0 \end{cases}.$$

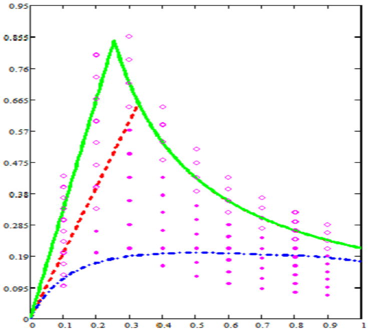

Fig.: the necessary condition, the criterion, the sufficient condition, stability testing

Conclusion

- The criterion (the necessary and sufficient condition) for weak conservativeness is deduced for the first time for the considered kind of schemes.
- The criterion turned out to be essentially wider than a sufficient condition of Y.V. Sheretov.
- Practitioners often confine themselves by necessary conditions only. In our case, the necessary spectral condition is too rough in the most interesting region of parameters and therefore it is not good enough for practical goals.
- Though derivation of the criterion is more complicated than that of the necessary condition, it is namely the criterion that the results of numerical experiments correspond well with.

Ačiū už dėmesį!

Thank you for your attention!