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Free algebras of Hilbert automorphic forms

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1 Definitions and notation

Let $d > 0$ be an integer squarefree number. Consider in pseudo-Euclidian space $(E^{2,2}(\mathbb{R}), (\ ,\ ))$ the lattice $L_d = U \oplus B_d$, where $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$B_d = \begin{cases} \begin{pmatrix} 2 & 1 \\ 1 & \frac{1-d}{2} \end{pmatrix}, \textit{if } d \equiv 1 \pmod 4; \\ \begin{pmatrix} 2 & 0 \\ 0 & -2d \end{pmatrix}, \textit{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Denote by D the connected component of the domain

$$\tilde{D} = \{[\omega] \in \mathbb{P}(E^{2,2}(\mathbb{C})); (\omega, \omega) = 0,\ (\omega, \overline{\omega}) > 0\}.$$

The domain D is 2-dimensional Hermitian symmetric space of type IV (according to Cartan classification). Let $O(L_d)$ be the group of isometries of lattice L_d and $\Gamma_d := O^+(L_d)$ – the index two subgroup in $O(L_d)$, preserving the domain D. Denote by $D^\bullet$ the cone over D in $E^{2,2}$.

Definition 1. *Automorphic form of weight k for Γ_d with character $\chi : \Gamma_d \to \mathbb{C}^*$ is a holomorphic function f in $D^\bullet$ such that:*
1) $f(tz) = t^{-k} f(z)$, $t \in \mathbb{C}^*$
2) $f(g(z)) = \chi(g) f(z)$, $g \in \Gamma_d$.

Definition 2. *Indivisible vector $e \in L_d$ is called a primitive root if $(e,e) < 0$ and the reflection in the subspace $e^\perp$ belongs to Γ_d. The length of the root e is the number (e,e). If $(e,e) = -2k$, we will speak about $-2k$-root e.*

Definition 3. *The Hirzebruch-Mumford volume of the orthogonal group $O^+(L)$ of the lattice L is a normalised covolume $\mathrm{Vol}_{HM}(D/O^+(L)) = \frac{\mathrm{Vol}(D/O^+(L))}{\mathrm{Vol}(D^c)}$, where D^c is a compact Hermitian symmetric space dual to the space D.*

We will use the notation $\mathrm{Vol}_{HM}(\Gamma)$ meaning the covolume $\mathrm{Vol}_{HM}(D/\Gamma)$.

Notation:
$A(\Gamma_d)$ – the algebra of automorphic forms for the group Γ_d with trivial character;
$\{e_i\}$ – the basis of lattice L_d;
$d(L)$ – the determinant of Gram matrix of the lattice L;
$\epsilon_p(L)$ – the Hasse index of the lattice L over $\mathbb{Z}_p$;
$\nu_2(L)$ – the maximal power of 2, dividing $d(L)$;
$d_L = \frac{d(L)}{2^{\nu_2(L)}}$;
$L^\vee$ – the dual lattice of the lattice L;
$disc(L) = L^\vee/L$ – the discriminant group of the lattice L, $disc(L)_{(2)}$ – 2-torsion elements in the group $disc(L)$;
p – prime number;
$\mathbb{Z}_p$ – p-adic integers, $\mathbb{Z}_p^*$ – invertible p-adic integers;

$(\cdot,\cdot)_p$ – p-adic Hilber symbol;
$L(s,d) = L(s,\chi) = \prod\limits_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$ – Dirichlet L-function with character $\chi(p) = \left(\frac{d}{p}\right)$;
$\rho(d)$ – the number of odd prime divisors of the number d;
$U(2) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$;
E – identity matrix;
$\mathcal{O}_{\mathcal{K}}$ – the ring of integers of the field $\mathcal{K} = \mathbb{Q}(\sqrt{d})$;
$\tilde{\Gamma}_d = \hat{SL}(2,\mathcal{O}) := Gal(\mathcal{K}/\mathbb{Q}) \ltimes SL(2,\mathcal{O}_{\mathcal{K}})$ – symmetric Hilbert modular group.
It is well-known that the group Γ_d is a maximal arithmetic subgroup of the group $O(2,2;\mathbb{R})$, containing symmetric Hilber modular group $\tilde{\Gamma}_d$ ([16],[5]).

2 Introduction

It is well-known, that if the algebra $A(\Gamma)$ is free, then the group Γ is generated by complex reflections. So we will consider only such subgroups Γ of the group Γ_d. Moreover, we suppose that $-E \in \Gamma$.

Remark 2.1. *There are no automorphic forms of an odd weight in the algebra* $A(\Gamma)$*.*

Доказательство. If f is an automorphic forms of an odd weight, it folows from the property 1) that $f(-z) = f(z)$, and from property 2) that $f(-z) = -f(z)$. □

The question of whether the algebra $A(\Gamma)$ is free is one of the main questions in transcendental invariant theory ([17]). By now there are several examples of groups Γ acting in domains D, that have free algebras of automorphic forms ([9], [7], [8], [17], [18]). The purpose of this paper is to find all d, for which the algebra $A(\Gamma)$ can be free. The following theorem gives the answer.

Theorem 1. *If the algebra $A(\Gamma)$ is free then $d \in \{2,\ 3,\ 5,\ 6,\ 13,\ 21\}$.*

Corollary 1. *The algebra $A(\Gamma_d)$ is free iff $d = 2$ or $d = 5$.*

□ The value of d is not 3 since as E. B. Vinberg told us, the group Γ_3 is not generated by reflections. All the rest values of d are discarded because of the necessary condition of the absence of singularity at the infinity ([12]). □

We will need one theorem of J. H. Bruinier ([3]). In the case of interest to us and in a form convenient for us it states the following:

Theorem A. *Let L be an even lattice of signature $(2, 2)$, such that the Witt index of the space $L \otimes \mathbb{Q}$ equals 1, and let Γ' be a finite index subgroup of the group $O^+(L)$. For each such subgroup we define the number $K(\Gamma')$:*

$$K(\Gamma') := \frac{\frac{1}{2}\sum\limits_{[\pi]} \mathrm{Vol}_{HM}(\Gamma'_\pi)}{\mathrm{Vol}_{HM}(\Gamma')}, \tag{1}$$

where Γ'_π is the stabiliser of the mirror π in the group Γ', the sum is taken over all Γ'-conjugacy classes of the mirrors of reflections π in the group Γ'. Let F be a meromorphic automorphic form for the group Γ' of weight K such that it has zeros on all mirrors of reflections in Γ' and only on them. If all the zeros are simple then $K(\Gamma') = K$.

Theorem B. [2] *If the algebra of* Γ'*-automorphic forms is free with the generators of weights* $k_1,\ k_2,\ k_3$*, then there exists unique up to the proportionality* Γ'*-automorphic form* F *of weight* $2 + k_1 + k_2 + k_3$ *(with non-trivial character) with simple zeros on all mirrors of reflections.*

If the algebra $A(\Gamma)$ is free then it has three generators of even weight. Hence the weight K of the form F form theorem B for $\Gamma' = \Gamma$ is not less then eight. Applying to the form F theorem A we get that $K(\Gamma) \geq 8$.

The following important proposition belongs to O. V. Shvartsman.

Proposition 2.1. $K(\Gamma_d) \geq K(\Gamma)$.

The idea of the proof of theorem 1 is to estimate in formula (1) with $\Gamma' = \Gamma_d$ the numerator from the above and the denominator from the below. We show that the denominator grows faster as a function of d, thus $K(\Gamma_d)$, and, hence, $K(\Gamma)$, is less then eight when d is great enough. From now on we will refer to the formula (1) meaning $\Gamma' = \Gamma_d$.

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3 The calculation of volumes

We use the formula for the computation of the volume $\mathrm{Vol}_{HM}(O^+(L))$ of given lattice L of signature $(2, n-2)$, which is unique in it's genus

([6], [14]):

$$\mathrm{Vol}_{HM}(O^+(L)) = 4|d(L)|^{\frac{n+1}{2}} \prod_{k=1}^{n} \pi^{-k/2}\Gamma(k/2) \cdot \prod_{p} a_p(L)^{-1}, \qquad (2)$$

where $a_p(L) := \frac{1}{2} \lim_{r\to\infty} p^{-\frac{rn(n-1)}{2}} |O(L \otimes \mathbb{Z}/p^r\mathbb{Z})|$ are local volumes of the lattice L. The formulas for computing the local volumes can be found, for example, in [10]. In order to prove the main theorem we prove the following lemmas.

Lemma 3.1. *a) If* $d \equiv 1 \pmod 4$ *then the volume* $\mathrm{Vol}_{HM}(\Gamma_d) = \frac{d^{3/2}}{2^{\rho(d)+4}\cdot 3\pi^2} L(2,d)$*;*
b) If $d \equiv 2,\ 3 \pmod 4$ *then the volume* $\mathrm{Vol}_{HM}(\Gamma_d) = \frac{d^{3/2}}{2^{\rho(d)+2}\cdot 3\pi^2} L(2,4d)$.

Lemma 3.2. *The volume* $\mathrm{Vol}_{HM}(O^+(L^e))$ *is not greater than* $C_1 \cdot \prod_{p|d_e} \frac{p+1}{2}$*, where* $C_1 = \frac{1}{12}$*, if* $\nu_2(L^e) = 1$*, and* $C_1 = \frac{1}{8}$*, if* $\nu_2(L^e) \in \{2,3,4\}$.

Lemma 3.3. *Let* I_e *be the set of odd prime numbers, dividing* d_e *such that for each* $p \in I_e$ *the lattice* $L^e \otimes \mathbb{Z}_p$ *is not isometric to any of the lattices* $U \oplus \langle 2d_e\rangle$, $U \oplus \langle 4d_e\rangle$, $U(2) \oplus \langle 2d_e\rangle$ *or* $U(2) \oplus \langle 4d_e\rangle$*. Then* $Vol(\Gamma^e) \le C_1 \cdot \prod_{p\in I_e} \frac{p-1}{2} \prod_{p\notin I_e} \frac{p+1}{2} = C_1 \cdot \prod_{p} \frac{p+1}{2} \prod_{p\in I_e} \frac{1-p^{-1}}{1+p^{-1}}$*, where* C_1 *is the same as in lemma 3.2.*

Lemma 3.4. *If* $d = 3 \pmod 4$ *and* e *is a primitive root of the lattice* L_d *such that* $\nu_2(\langle e\rangle) = 1$, $\nu_2(L^e) = 3$ *and* $d_e = 3 \pmod 4$*, then* $\mathrm{Vol}_{HM}(\Gamma^e) \le \frac{1}{24} \prod_{p|d_e} \frac{p+1}{2}$.

Let $d = p_1 \cdot \ldots \cdot p_{\rho(d)}$ or $d = 2p_1 \cdot \ldots \cdot p_{\rho(d)} = 2d'$, where p_i are different odd prime numbers, $3 \le p_1 < \ldots < p_{\rho(d)}$ (if $d = 2d'$, the set $\{p_i\}_i$ might be empty).

Lemma 3.5. *The denominator of the formula (1) is not less than* $C\cdot d^{\frac{3}{2}}$, *where* $C = \frac{1}{2^{\rho(d)+4}\cdot 45}$ *if* $d \equiv 1 \pmod 4$, *and* $C = \frac{1}{2^{\rho(d)+2}\cdot 45}$ *if* $d \equiv 2 \pmod 4$ *or* $d \equiv 3 \pmod 4$.

Lemma 3.6. *The numerator of the formula (1) is not greater than* $C_2 \cdot \prod_{i=1}^{\rho(d)} \frac{p_i+3}{2}$, $C_2 = \frac{1}{24}$ *if* $d \equiv 1 \pmod 4$, $C_2 = \frac{1}{6}$ *if* $d \equiv 3 \pmod 4$ *and* $C_2 = \frac{7}{24}$ *if* $d \equiv 2 \pmod 4$.

Denote by $\tilde{K}(\Gamma_d)$ the estimate from the above on the value $K(\Gamma_d)$, recieved in lemmas 3.5 and 3.6. It is easy to check the following facts.

Remark 3.1. *If* $\tilde{K}(\Gamma_d) < 8$ *for* $k = k_0$ *for each set of odd prime numbers* $(p_1, \dots, p_{k_0})$, *then for each* $k > k_0$ *and each set of odd prime numbers* $(p_1, \dots, p_k)$ *the inequality is preserved.*

Remark 3.2. *If* $\tilde{K}(\Gamma_d) < 8$ *on the set* $(p_1, \dots, p_m)$, *then if we replace one of* p_i *on* $p_j > p_i$,*the inequality is preserved.*

The proof of theorem 1 is straightforward. Assume d is odd. It follows from lemmas 3.5 and 3.6 that $K(\Gamma_d) \leq \tilde{K}(\Gamma_d) = \frac{30\prod_{i=1}^{\rho(d)}(p_i+3)}{\prod_{i=1}^{\rho(d)} p_i^{3/2}}$. We consider the sets of k primes in the ascending order. It is easy to check that the inequality $K(\Gamma_d) \geq 8$ provides us a finite list of values of d for which the corresponding algebras $A(\Gamma_d)$. If d is even one can check analogously to the case of odd d that $\tilde{K}(\Gamma_d) > 8$ only for finitely many values of d.

We notice that instead of the estimate of the denominator of formula (1) from lemma 3.5, which is convenient for the calculation in general, one could use the exact value, computed in lemma 3.1. Substituting the values of L-functions $L(2,d)$ or $L(2,4d)$ to the denominator for all the rest d, we get that $\tilde{K}(\Gamma_d) < 8$ for all

$$d \notin \{35, 30, 15, 11, 10, 7, 21, 13, 6, 5, 3, 2\}$$

. Then we show that $K(\Gamma_d) < 8$ if $d \in \{35,\ 30,\ 15,\ 11,\ 10,\ 7\}$ using detailed analysis of existing reflections in each case and calculating corresponding covolumes, which concludes the proof.

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